

# CONTACT OF A THIN FREE BOUNDARY WITH A FIXED ONE IN THE SIGNORINI PROBLEM

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**ABSTRACT.** We study the Signorini problem near a fixed boundary, where the solution is “clamped down” or “glued.” We show that in general the solutions are at least  $C^{1/2}$  regular and that this regularity is sharp. We prove that near the actual points of contact of the free boundary with the fixed one the blowup solutions must have homogeneity  $\kappa \geq 3/2$ , while at the non-contact points the homogeneity must take one of the values:  $1/2, 3/2, \dots, m - 1/2, \dots$

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. The Signorini problem.** The purpose of this paper is to study the behavior of the thin free boundary as it approaches to the fixed boundary in the so-called (scalar) *Signorini problem* (also known as the *thin obstacle problem*).

The Signorini problem consists in minimizing the Dirichlet energy functional

$$(1.1) \quad J(v) := \int_{B_1^+} |\nabla v|^2$$

on a closed convex set

$$(1.2) \quad \mathcal{K} = \mathcal{K}(g) := \{v \in W^{1,2}(B_1^+) : v = g \text{ on } (\partial B_1)^+, v \geq 0 \text{ on } B_1'\},$$

for a given function  $g \in L^2((\partial B_1)^+)$ . Here and everywhere in the paper we use the following notations:

$$B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}, \quad B_r := B_r(0), \\ E^+ := E \cap \{x_n > 0\}, \quad E' := E \cap \{x_n = 0\},$$

for a subset  $E \subset \mathbb{R}^n$ . We assume  $n \geq 2$ . Using direct methods of calculus of variation one can verify that a minimizer  $u \in \mathcal{K}$  exists and satisfies the following variational inequality:

$$(1.3) \quad \int_{B_1^+} \nabla u \nabla (v - u) \geq 0 \quad \text{for any } v \in \mathcal{K}.$$

The problem above goes back to the foundational paper [LS67] on variational inequalities. It has been known for quite some time that the minimizers are in

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the class  $C^{1,\alpha}(B_1^+ \cup B_1')$  for some  $\alpha > 0$  (see [Caf79] and also [Ura87]) and even  $C^{1,1/2}(B_1^+ \cup B_1')$  in the dimension  $n = 2$ , see [Ric78]. Besides, the minimizers satisfy

$$\begin{aligned} \Delta u &= 0 \quad \text{in } B_1^+ \\ u &\geq 0, \quad -\partial_{x_n} u \geq 0, \quad u \partial_{x_n} u = 0 \quad \text{on } B_1'. \end{aligned}$$

The latter are known as the *Signorini* or *complementarity boundary conditions*.

The problem features the following apriori unknown subsets of  $B_1'$ :

$$\begin{aligned} \Lambda(u) &:= \{x \in B_1' : u = 0\} && \text{the coincidence set} \\ \Omega(u) &:= \{x \in B_1' : u > 0\} && \text{the non-coincidence set} \\ \Gamma(u) &:= \partial_{B_1'} \Omega(u) && \text{the free boundary.} \end{aligned}$$

The study of the geometric and analytic properties of the free boundary is one of the objectives of the Signorini problem. Sometimes it is said that the free boundary  $\Gamma(u) \subset B_1'$  is *thin*, to indicate that it is (expected to be) of dimension  $(n - 2)$ .

Recent years have seen some interesting new developments in the problem, starting with the proof in [AC04] that the minimizers  $u$  are in the class  $C^{1,1/2}(B_1^+ \cup B_1')$ , in any dimension  $n \geq 2$ , which is the optimal regularity. This opened up the possibility of studying the free boundary  $\Gamma(u)$ , which has been done in [ACS08, CSS08, GP09], see also [PSU12, Chapter 9]. An effective tool in the study of the free boundary is *Almgren's frequency formula*

$$N^x(r, u) := \frac{r \int_{B_r^+(x)} |\nabla u|^2}{\int_{(\partial B_r)^+} u^2}.$$

It originated in the work of Almgren on multi-valued harmonic functions [Alm00] and has an important property of being monotone in  $r$ , even for solutions of the Signorini problem. One then classifies the free boundary points according to the value

$$\kappa := N^x(0+, u).$$

It is known that  $\kappa \geq 3/2$  for  $x \in \Gamma(u)$  in the Signorini problem and more precisely  $\kappa = 3/2$  or  $\kappa \geq 2$  [ACS08]. This results in a decomposition

$$\Gamma(u) = \Gamma_{3/2}(u) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(u), \quad \text{where } \Gamma_\kappa(u) := \{x \in \Gamma(u) : N^x(0+, u) = \kappa\}.$$

The set  $\Gamma_{3/2}(u)$  is known as the *regular set*. It has been recently shown that  $\Gamma_{3/2}(u)$  is real analytic [KPS14] by using a partial hodograph-Legendre transform from  $C^{1,\alpha}$  regularity proved in [ACS08]. See also [DSS14b] for a different proof of  $C^\infty$  regularity, based on a generalization of the boundary Harnack principle. The only other free boundary points studied in the literature are the ones in  $\Gamma_{2m}(u)$ ,  $m \in \mathbb{N}$  which correspond to the points where the coincidence set  $\Lambda(u)$  has a zero  $H^{n-1}$  density, see [GP09]. Such points are known as *singular points*. It was proved in [GP09] that  $\Gamma_{2m}(u)$  is contained in a countable union of  $C^1$  manifolds.

An interesting question is finding all possible values for  $\kappa = N^x(0+, u)$ . In dimension  $n = 2$  the answer to that question is known (proof is a simple exercise):  $\kappa$  must be one of the following values:

$$3/2, 2, 7/2, 4, \dots, 2m - 1/2, 2m, \dots$$

However, this is still an open problem in dimensions  $n \geq 3$ .

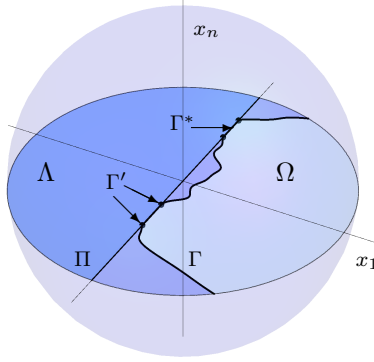


FIGURE 1. Free boundary  $\Gamma$  near the contact points  $\Gamma'$  with the fixed boundary  $\Pi$  considered in the hyperplane  $\{x_n = 0\}$ .

**1.2. Contact of the free and fixed boundaries.** The objective in this paper is the study of the behavior of the free boundary  $\Gamma(u)$  in the Signorini problem as it approaches a set where  $u$  is forced to be zero. More precisely, consider a closed subset  $\mathcal{K}_0$  of the set  $\mathcal{K}$  in (1.2), defined by

$$(1.4) \quad \mathcal{K}_0 = \mathcal{K}_0(g) := \{v \in \mathcal{K}(g) : v = 0 \text{ on } B'_1 \cap \{x_1 \leq 0\}\}$$

and minimize the Dirichlet energy  $J$  in (1.1) over  $\mathcal{K}_0$ . That is, compared to the Signorini problem, we have an additional constraint that the functions must vanish on  $B'_1 \cap \{x_1 \leq 0\}$ . If we think of the solution of the Signorini problem as an elastic membrane that is forced to stay above zero in  $B'_1$ , the new constraint in  $\mathcal{K}_0$  can be thought of as “clamping down” or “gluing” the membrane on  $B'_1 \cap \{x_1 \leq 0\}$ . The boundary of the latter set in  $B'_1$  is

$$\Pi := \{x_1 = 0, x_n = 0\},$$

which we call the *fixed boundary*. Note that the coincidence set  $\Lambda(u)$  will contain now  $B'_1 \cap \{x_1 \leq 0\}$  and the truly free part of  $\Gamma(u)$  is  $\Gamma(u) \cap \{x_1 > 0\}$ . The points in

$$\Gamma'(u) := \overline{\Gamma(u) \cap \{x_1 > 0\}} \cap \Pi$$

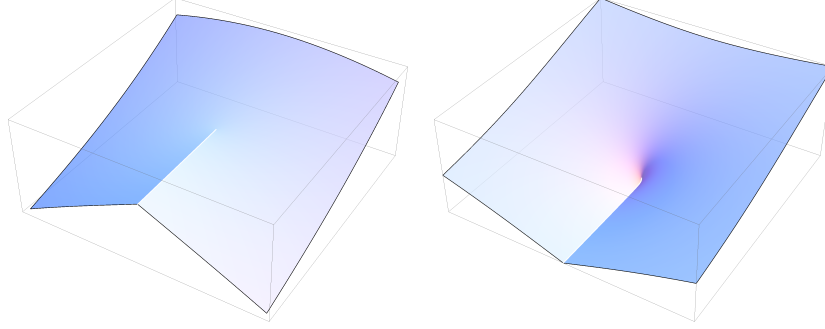
are categorized as *contact points*, and the ones in

$$\Gamma^*(u) := (\Gamma(u) \cap \Pi) \setminus \Gamma'(u)$$

are *non-contact points*, see Fig. 1. We note that the minimizers in  $\mathcal{K}_0$  still solve the Signorini problem in small halfballs  $B_r^+(x_0)$  with  $x_0 \in B'_1 \cap \{x_1 > 0\}$  and therefore we will have that  $u \in C_{\text{loc}}^{1,1/2}(B_1^+ \cup (B'_1 \cap \{x_1 > 0\}))$  and that it satisfies

$$\begin{aligned} \Delta u &= 0 \quad \text{in } B_1^+ \\ u &= 0 \quad \text{on } B'_1 \cap \{x_1 \leq 0\} \\ u &\geq 0, \quad -\partial_{x_n} u \geq 0, \quad u \partial_{x_n} u = 0 \quad \text{on } B'_1 \cap \{x_1 > 0\}. \end{aligned}$$

There are many papers in the literature dealing with the contact of the free and fixed boundaries in various free boundary problems. The case of the classical obstacle problem, for instance, was studied by [Ura96, AU95]. We also refer to [PSU12,



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FIGURE 2. Examples of solutions limiting the optimal regularity:  $\hat{u}_{3/2}(x)$  is an explicit solution of the Signorini problem and  $\hat{u}_{1/2}(x)$  is a minimizer over  $\mathcal{K}_0$  with worst possible regularity.

Chapter 8] and references therein for some of these results, including also extensions to other obstacle-type problems.

In contrast to the case of the classical obstacle problem, where the presence of the fixed boundary actually helps – for instance, to avoid a geometric “thickness” condition on coincidence set needed for the regularity of the free boundary – in the Signorini problem the presence of the fixed boundary introduces a serious handicap. Indeed, as we have mentioned earlier, the optimal regularity of the Signorini problem is  $C^{1,1/2}$ . This regularity is exhibited by the following explicit solution:

$$(1.5) \quad \hat{u}_{3/2}(x) := \operatorname{Re}(x_1 + i|x_n|)^{3/2}.$$

On the other hand, it is easy to see that

$$(1.6) \quad \hat{u}_{1/2}(x) := \operatorname{Re}(x_1 + i|x_n|)^{1/2}$$

is a minimizer of  $J$  over  $\mathcal{K}_0$  (simply because it is harmonic in  $B_1 \setminus (B'_1 \cap \{x_1 \leq 0\})$ ), thus limiting the generally expected regularity of minimizers of  $J$  to at most  $C^{1/2}$ . (See Fig. 2 for the illustration of these solutions.)

This lower regularity of minimizers undercuts many techniques used for the Signorini problem, calling for caution even when dealing with the first derivatives of the solution. Luckily, however, one of the most important tools in our analysis, Almgren’s frequency formula, still works: one of the steps in the proof is based on a Rellich-type identity, which in our case becomes an inequality in the correct direction and allows the proof to go through.

**1.3. Main results.** The first main result in this paper establishes the optimal regularity of the minimizers.

**Theorem 1.1** (Optimal regularity). *If  $u$  is a minimizer of the functional  $J$  in (1.1) over  $\mathcal{K}_0$  in (1.4), then  $u \in C_{\text{loc}}^{1/2}(B_1^+ \cup B'_1)$  with*

$$\|u\|_{C^{1/2}(B_{1/2}^+ \cup B'_{1/2})} \leq C_n \|u\|_{L^2(B_1^+)}.$$

The regularity above implies that for any  $x \in \Gamma(u)$  we have

$$\kappa = N^x(0+, u) \geq 1/2.$$

The knowledge of the possible values of  $\kappa$  is important for the classification of free boundary points (as we discussed at the end of subsection 1.1). Concerning these values we have the following results.

**Theorem 1.2** (Minimal Almgren's frequency at contact points). *If  $u$  is a minimizer of  $J$  over  $\mathcal{K}_0$ , then for a contact point  $\bar{x} \in \Gamma'(u)$  we have*

$$\kappa = N^{\bar{x}}(0+, u) \geq 3/2.$$

At non-contact points we give a more complete picture.

**Theorem 1.3** (Almgren's frequency at non-contact points). *If  $u$  is a minimizer of  $J$  over  $\mathcal{K}_0$ , then for a non-contact point  $\bar{x} \in \Gamma^*(u)$  we have that*

$$\kappa = N^{\bar{x}}(0+, u)$$

*can take only the following values:*

$$1/2, 3/2, 5/2, \dots, m - 1/2, \dots$$

## 2. OPTIMAL REGULARITY

**2.1. Symmetrization.** It will be convenient for our considerations to extend every function  $v \in \mathcal{K}_0$  by even symmetry in  $x_n$ -variable to the entire ball  $B_1$ :

$$v(x', -x_n) := v(x', x_n) \quad \text{for } (x', x_n) \in B_1^+.$$

With such extension in mind, the energy  $J$  in (1.1) can be replaced with

$$(2.1) \quad J(v) := \frac{1}{2} \int_{B_1} |\nabla v|^2.$$

**2.2. Hölder continuity.** As the first result towards the optimal regularity, we show that the minimizers are  $C^\alpha$  regular for some  $\alpha > 0$ .

**Proposition 2.1** (Hölder continuity). *If  $u$  is a minimizer of  $J$  over  $\mathcal{K}_0$ , then  $u \in C^\alpha(B_{1/2})$ , with a dimensional constant  $\alpha > 0$  and*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C_n \|u\|_{L^2(B_1)}.$$

We start by showing that the positive and negative parts of the minimizer  $u$  are subharmonic. Note that at this stage we have not yet established the continuity of  $u$ , so we will resort to the energy methods.

**Lemma 2.2.**  $u_\pm = \max\{\pm u, 0\}$  are subharmonic functions in  $B_1$ .

*Proof.* Proving the lemma is equivalent to showing that for any nonnegative test function  $\eta \in C_0^\infty(B_1)$  we have

$$(2.2) \quad \int_{B_1} \nabla u_\pm \nabla \eta \leq 0.$$

Let  $\psi_\varepsilon \in C^\infty(\mathbb{R})$  be a nondecreasing function such that

$$\psi_\varepsilon = 0 \quad \text{in } (-\infty, \varepsilon), \quad 0 \leq \psi_\varepsilon \leq 1 \quad \text{in } (\varepsilon, 2\varepsilon), \quad \psi_\varepsilon = 1 \quad \text{in } (2\varepsilon, \infty).$$

Then for a fixed  $\varepsilon > 0$  and sufficiently small  $|t|$  we have

$$\begin{aligned} \{u > 0\} &= \{u + t\eta\psi_\varepsilon(u_\pm) > 0\} \\ \{u < 0\} &= \{u + t\eta\psi_\varepsilon(u_\pm) < 0\} \end{aligned}$$

and thus  $u + t\eta\psi_\varepsilon(u_\pm)$  are admissible functions from  $\mathcal{K}_0$ . Since  $u$  is a minimizer, we have  $J(u + t\eta\psi_\varepsilon(u_\pm)) \geq J(u)$ , yielding

$$0 = \int_{B_1} \nabla u \nabla (\eta\psi_\varepsilon(u_\pm)) = \int_{B_1} \nabla u \nabla \eta\psi_\varepsilon(u_\pm) \pm \int_{B_1} |\nabla u|^2 \psi'_\varepsilon(u_\pm) \eta.$$

Since the second integral is nonnegative, sending  $\varepsilon$  to 0 we obtain (2.2).  $\square$

Once we know that  $u_\pm$  are subharmonic in  $B_1$ , we immediately obtain that  $u$  is locally bounded.

**Lemma 2.3** (Local boundedness). *If  $u$  is a minimizer of  $J$  over  $\mathcal{K}_0$ , then  $u \in L^\infty(B_{3/4})$  and more precisely*

$$\sup_{B_{3/4}} |u| \leq C_n \|u\|_{L^2(B_1)}. \quad \square$$

We can now proceed to the proof of Hölder continuity.

*Proof of Proposition 2.1.* Using the local boundedness and the fact that  $u_\pm$  vanish on  $B'_1 \cap \{x_1 \leq 0\}$ , by the comparison principle we can write that

$$(2.3) \quad |u| \leq Mv \quad \text{in } B_{3/4},$$

where  $M = C_n \|u\|_{L^2(B_1)}$  and  $v$  solves

$$(2.4) \quad \begin{aligned} \Delta v &= 0 \quad \text{in } B_{3/4} \setminus (B'_{3/4} \cap \{x_1 \leq 0\}) \\ v &= 0 \quad \text{on } B'_{5/8} \cap \{x_1 \leq 0\} \\ v &= 1 \quad \text{on } \partial B_{3/4} \end{aligned}$$

with boundary values changing continuously from 0 to 1 in  $(B'_{3/4} \setminus B'_{5/8}) \cap \{x_1 \leq 0\}$ . We next claim that the barrier function  $v$  above is in  $C^\alpha(B_{1/2})$ . Indeed, we can use a bi-Lipschitz transformation to map  $B_{3/4} \setminus (B'_{3/4} \cap \{x_1 \leq 0\})$  to  $B_{3/4}^+$  preserving the distance from the origin. Then  $v$  will transform into  $w$ , which would be a solution of a uniformly elliptic equation in divergence form with measurable coefficients:

$$(2.5) \quad \begin{aligned} \operatorname{div}(a_{ij}w_j) &= 0 \quad \text{in } B_{3/4}^+ \\ w &= 0 \quad \text{on } B'_{5/8}. \end{aligned}$$

By the De Giorgi-Nash theorem, we know  $w \in C^\alpha(B_{1/2}^+)$ , and since the transformation is bi-Lipschitz we also get  $v \in C^\alpha(B_{1/2})$ , which provides

$$(2.6) \quad |v(x)| \leq C \operatorname{dist}(x, B'_1 \cap \{x_1 \leq 0\})^\alpha.$$

The latter, together with (2.3) gives

$$(2.7) \quad |u(x)| \leq CM \operatorname{dist}(x, B'_1 \cap \{x_1 \leq 0\})^\alpha.$$

Combined with the next lemma, this implies  $u \in C^\alpha(B_{1/2})$ .  $\square$

**Lemma 2.4.** *Let  $u$  be a minimizer of  $J$  over  $\mathcal{K}_0$ . If for a  $0 < \beta \leq 1$  and all  $x, y \in B_{1/2}$  the following property holds:*

$$(2.8) \quad |u(x)| \leq C_0 \operatorname{dist}(x, B'_1 \cap \{x_1 \leq 0\})^\beta$$

*then  $u \in C^\beta(B_{1/2})$  with  $\|u\|_{C^\beta(B_{1/2})}$  depending only on  $C_0, n, \beta$ .*

*Proof.* Denote  $d_x := \operatorname{dist}(x, B'_1 \cap \{x_1 \leq 0\})$ . Take any  $x, y \in B_{1/2}$ . Without loss of generality we can assume  $x \in B_1^+$  and  $d_y \leq d_x$ . We will consider three cases:

1)  $|x - y| > d_x/8$ . Using (2.8) we get

$$|u(x) - u(y)| \leq C_0(d_x^\beta + d_y^\beta) \leq 2C_0 8^\beta |x - y|^\beta.$$

2)  $|x - y| \leq d_x/8$  and the  $n$ -th coordinate of  $x$ ,  $x_n > d_x/4$ . In this case we observe that  $B_{d_x/4}(x) \subset B_1^+$  and thus  $u$  is harmonic there,  $x, y \in B_{d_x/8}(x)$  and the interior gradient estimates for harmonic functions imply

$$\begin{aligned} |u(x) - u(y)| &\leq C_n \|u\|_{L^\infty(B_{d_x/4}(x))} \frac{|x - y|}{d_x} \\ &\leq C_n C_0 (5/4)^\beta d_x^\beta \frac{|x - y|^\beta (d_x/8)^{1-\beta}}{d_x} = C |x - y|^\beta. \end{aligned}$$

3)  $|x - y| \leq d_x/8$  and  $x_n \leq d_x/4$ . In this case  $B_{(3/4)d_x}(x', 0) \subset B_{d_x}(x)$ . Thus  $u$  solves the Signorini problem in  $B_{(3/4)d_x}(x', 0)$  and  $x, y \in B_{(3/8)d_x}(x', 0)$ . Using the interior Lipschitz regularity for the solutions of the Signorini problem, see [AC04, Theorem 1], we have

$$|u(x) - u(y)| \leq C_n \|u\|_{L^\infty(B_{(3/4)d_x}(x))} \frac{|x - y|}{d_x}$$

and we complete the proof as in the previous case.  $\square$

**2.3. Monotonicity formula in the halfball.** As we observed in the introduction, we know that the function  $\hat{u}_{1/2}$  restricts the regularity of our solutions to  $C^{1/2}$ . In order to rigorously obtain that  $C^{1/2}$  is also the minimum expected (and thus optimal) regularity, we need the following monotonicity formula for the halfball, first introduced in [AC04].

**Lemma 2.5** (Monotonicity formula, [AC04, Lemma 4]). *For any  $w \in C(\overline{B_1^+})$  satisfying*

$$\begin{aligned} \Delta w &= 0 && \text{in } B_1^+, \\ w &= 0 && \text{on } B_1' \cap \{x_1 \leq 0\}, \\ w &\geq 0, \quad w \partial_{x_n} w = 0 && \text{on } B_1'. \end{aligned}$$

*Then the function*

$$\varphi(r) := \frac{1}{r} \int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}} dx$$

*is nondecreasing for  $r \in (0, 1)$ .*

*Proof.* The proof is a verbatim repetition of that of [AC04, Lemma 4], despite of the slight difference in the assumptions. Namely, instead of asking the convexity of the set  $\{x' \in B_1' : w(x', 0) > 0\}$ , we note that it is only used to show that the complement set of the support of  $w$  contains the lower dimensional halfball  $B_1' \cap \{x_1 \leq 0\}$ , which is automatically satisfied in the setting of our problem.  $\square$

**2.4. Optimal  $C^{1/2}$  regularity of minimizers.** We are now ready to proof our first main result.

*Proof of Theorem 1.1.* We apply the monotonicity formula in Lemma 2.5 to the minimizer  $u$  of  $J$  to obtain

$$(2.9) \quad \varphi(r) \leq \varphi(3/4) \leq C \|u\|_{L^2(B_1)}^2.$$

Here, the last inequality is standard for non-negative subharmonic functions (for a proof see for example [Caf98]). Applying this for  $u_{\pm}$  we obtain the corresponding inequality for  $u$ .

Now using the fact that  $u$  vanishes on  $B'_1 \cap \{x_1 \leq 0\}$  we also have the Poincaré inequality for the halfball

$$(2.10) \quad \int_{B_r^+} u^2 \leq C_n r^2 \int_{B_r^+} |\nabla u|^2.$$

Then by the scaling of Lemma 2.3 we have

$$(2.11) \quad \begin{aligned} \sup_{B_{r/2}} |u| &\leq C_n r^{-\frac{n}{2}} \|u\|_{L^2(B_r)} \leq C_n r^{1-\frac{n}{2}} \|\nabla u\|_{L^2(B_r^+)} \\ &\leq C_n r^{\frac{1}{2}} \left( \frac{1}{r} \int_{B_r^+} \frac{|\nabla u|^2}{|x|^{n-2}} dx \right)^{1/2} \leq C_n r^{\frac{1}{2}} \varphi(r)^{1/2} \leq C_n r^{\frac{1}{2}} \|u\|_{L^2(B_1)}. \end{aligned}$$

Let us notice that the above estimate holds also for any ball  $B_{r/2}(x)$  with a center  $x \in B'_{1/2} \cap \{x_1 \leq 0\}$ , and  $r \leq 1/4$

$$(2.12) \quad \sup_{B_{r/2}(x)} |u| \leq C_n r^{\frac{1}{2}} \|u\|_{L^2(B_1)} \leq C_n r^{\frac{1}{2}} \|u\|_{L^\infty(B_1)}$$

yielding

$$(2.13) \quad |u| \leq C \operatorname{dist}(x, B'_1 \cap \{x_1 \leq 0\})^{1/2}.$$

Using Lemma 2.4 we obtain  $u \in C^{1/2}(B_{1/4})$ .  $\square$

*Remark 2.6.* Without loss of generality we will further assume that  $u \in C^{1/2}(B_1)$ .

### 3. MONOTONICITY OF THE FREQUENCY

**3.1. Almgren's Frequency Formula.** As we mentioned in the introduction, Almgren's frequency formula plays an important role in the Signorini problem. Since we have an additional constraint for functions in  $\mathcal{K}_0$ , it is not automatic that it will still be monotone. Fortunately, however, it is still the case.

**Theorem 3.1** (Monotonicity of the frequency). *If  $u$  is a minimizer of  $J$  over  $\mathcal{K}_0$ , then*

$$(3.1) \quad N(r) = N^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2}$$

*is monotone in  $r$  for  $r \in (0, R)$  and  $x_0 \in B'_1 \cap \{x_1 \geq 0\}$  such that  $B_R(x_0) \subset B_1$ . Moreover,  $N^{x_0}(r, u) \equiv \kappa$  for all  $0 < r \leq R$  iff  $u$  is homogeneous of degree  $\kappa$  in  $B_R(x_0)$ , with respect to the center  $x_0$ .*

The following notations will be used in the proof:

$$D(r) := \int_{B_r(x_0)} |\nabla u|^2 \quad \text{and} \quad H(r) := \int_{\partial B_r(x_0)} u^2.$$

Now if we consider the logarithm of  $N(r)$  and formally differentiate it, we obtain

$$\frac{N'(r)}{N(r)} = (\log N(r))' = \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}.$$



In order to prove the theorem, we need to show that the right hand side is non-negative. We accomplish this by proving differentiation formulas/inequalities in Lemmas 3.2, 3.3, and 3.4, following similar proofs in [GP09] or [ACS08].

We start with the following alternate formula for  $D(r)$ .

**Lemma 3.2** (First identity). *For the minimizers  $u$  of  $J$  over  $\mathcal{K}_0$ , the following identity holds for  $B_r(x_0) \Subset B_1$  with  $x_0 \in B'_1$ :*

$$(3.2) \quad D(r) = \int_{B_r(x_0)} |\nabla u|^2 = \int_{\partial B_r(x_0)} uu_\nu.$$

*Proof.* To prove the lemma we note that for any test function  $\eta \in W^{1,2}(B_r(x_0))$  which vanishes in a neighborhood of  $B'_1 \cap \{x_1 \leq 0\}$  then we have

$$(3.3) \quad \int_{B_r^+(x_0)} \nabla u \nabla \eta = \int_{B_r^+(x_0)} u_\nu \eta + \int_{(\partial B_r(x_0))^+} u_\nu \eta.$$

For a small  $\varepsilon > 0$ , choose  $\eta_\varepsilon(x) = u\psi(d(x)/\varepsilon)$ , where  $d(x) = \text{dist}(x, B'_1 \cap \{x_1 \leq 0\})$  and  $\psi \in C^\infty(\mathbb{R})$  is such that

$$\begin{aligned} \psi &= 0 \quad \text{in } (-\infty, 1), \quad 0 \leq \psi \leq 1 \quad \text{in } (1, 2), \quad \psi = 1 \quad \text{in } (2, \infty), \\ 0 &\leq \psi' \leq M \quad \text{in } (-\infty, \infty). \end{aligned}$$

We want to plug  $\eta = \eta_\varepsilon$  into (3.3) and let  $\varepsilon \rightarrow 0$ . We first claim that

$$(3.4) \quad \lim_{\varepsilon \rightarrow 0} \int_{B_r^+(x_0)} \nabla u \nabla \eta_\varepsilon = \int_{B_r^+(x_0)} |\nabla u|^2,$$

which is the same as

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0} \int_{B_r^+(x_0)} \nabla u (\nabla \eta_\varepsilon - \nabla u) = 0.$$

Indeed

$$\begin{aligned} \nabla \eta_\varepsilon &= \psi \left( \frac{d}{\varepsilon} \right) \nabla u + u \psi' \left( \frac{d}{\varepsilon} \right) \frac{\nabla d}{\varepsilon}, \\ \nabla \eta_\varepsilon - \nabla u &= \left( \psi \left( \frac{d}{\varepsilon} \right) - 1 \right) \nabla u + \frac{u}{\varepsilon} \psi' \left( \frac{d}{\varepsilon} \right) \nabla d. \end{aligned}$$

Multiplying both sides of the above by  $\nabla u$  and integrating over  $B_1$ , we obtain

$$(3.6) \quad \left| \int_{B_r^+(x_0)} \nabla u (\nabla \eta_\varepsilon - \nabla u) \right| \leq \int_{\{d \leq 2\varepsilon\}} |\nabla u|^2 + \frac{M}{\varepsilon} \int_{\{d \leq 2\varepsilon\}} u |\nabla u|,$$

using that  $|\psi'| \leq M$  and  $|\nabla d| \leq 1$ . Since the first integral on the right hand side goes to 0 as  $\varepsilon \rightarrow 0$ , it remains only to estimate the second one. We have

$$(3.7) \quad \frac{M}{\varepsilon} \int_{\{d \leq 2\varepsilon\}} u |\nabla u| \leq \left( \int_{\{d \leq 2\varepsilon\}} |\nabla u|^2 \right)^{1/2} \frac{M}{\varepsilon} \left( \int_{\{d \leq 2\varepsilon\}} u^2 \right)^{1/2}.$$

Again the first integral goes to 0, and to estimate the second one we use the  $C^{1/2}$  regularity of  $u$  to obtain

$$u^2 \leq C\varepsilon \quad \text{in } \{d \leq 2\varepsilon\}.$$

Besides, we also have that  $|\{d \leq 2\varepsilon\}| \leq C\varepsilon$ , which gives

$$\frac{1}{\varepsilon} \left( \int_{\{d \leq 2\varepsilon\}} u^2 \right)^{1/2} \leq C$$

and establishes (3.4). Now, to complete the proof of the lemma, we let  $\eta = \eta_\varepsilon$  in (3.3) and pass to the limit as  $\varepsilon \rightarrow 0$ . Using the fact that

$$u_\nu \eta_\varepsilon = u_\nu u \psi = 0 \quad \text{on } B'_1$$

we obtain (3.2).  $\square$

**Lemma 3.3** (Second identity). *For the minimizer  $u$  of  $J$  over  $\mathcal{K}_0$  the following identity holds for  $B_r(x_0) \Subset B_1$  with  $x_0 \in B'_1$ :*

$$(3.8) \quad H'(r) = \frac{n-1}{r} H(r) + 2 \int_{\partial B_r(x_0)} u u_\nu.$$

The differentiation formula should be understood in the sense that  $H(r)$  is an absolutely continuous function of  $r$  and that the differentiation formula holds for a.e.  $r$ .

*Proof.* We have

$$\begin{aligned} H(r) &= 2 \int_{(\partial B_r(x_0))^+} u^2 = 2 \int_{(\partial B_r(x_0))^+} \left( \frac{x - x_0}{r} \cdot \nu \right) u^2 \\ &= \frac{2}{r} \int_{B_r^+(x_0)} \operatorname{div}((x - x_0)u^2) \\ &= \frac{1}{r} \int_{B_r(x_0)} \operatorname{div}(x - x_0)u^2 + \frac{2}{r} \int_{B_r(x_0)} (x - x_0)(\nabla u)u. \end{aligned}$$

Hence, we obtain

$$H'(r) = \frac{n}{r} \int_{\partial B_r(x_0)} u^2 + \frac{2}{r} \int_{\partial B_r(x_0)} (x - x_0)(\nabla u)u - \frac{1}{r} H(r),$$

which yields the desired identity.  $\square$

While the above two identities were the same as in the Signorini problem, the third one becomes actually an inequality, which suffices for our purposes.

**Lemma 3.4** (Third (Rellich-type) inequality). *For the minimizer  $u$  of  $J$  over  $\mathcal{K}_0$  the following inequality holds for  $B_r(x_0) \Subset B_1$  with  $x_0 \in B'_1 \cap \{x_1 \geq 0\}$ :*

$$(3.9) \quad D'(r) \geq \frac{n-2}{r} D(r) + 2 \int_{\partial B_r(x_0)} u_\nu^2$$

or, equivalently,

$$(3.10) \quad r \int_{\partial B_r(x_0)} |\nabla u|^2 \geq \int_{B_r(x_0)} (n-2)|\nabla u|^2 + 2r \int_{\partial B_r(x_0)} u_\nu^2.$$

We explicitly observe that the center  $x_0$  of the ball  $B_r(x_0)$  must be in the upper thin halfball  $B'_1 \cap \{x_1 \geq 0\}$  for the inequality to hold.

*Proof.* The proof of this lemma uses the domain variation in radial direction similar to the one in [Wei98, p. 444]. The main difference is that our constraints allow us to make perturbations that increase the distance from the origin, thus yielding an inequality (with the correct sign) instead of the equality in the non-constrained case. We consider the function

$$\eta_k(y) := \max \left\{ 0, \min \left\{ 1, \frac{r - |y|}{k} \right\} \right\}.$$

Then for  $\varepsilon > 0$ , we have

$$u_\varepsilon(x) = u(x + \varepsilon \eta_k(x - x_0)(x - x_0)) \in \mathcal{K}_0.$$

Note that the same will not be true for negative  $\varepsilon$  (which is why we only have an inequality), that variation will bring over the zero values of  $u$  from  $B'_1 \cup \{x_1 \leq 0\}$  into  $B'_1 \cup \{x_1 > 0\}$ , rendering the variation not an admissible function. Once we established the admissibility of  $u_\varepsilon$ , we can translate  $x_0$  into the origin and continue the rest of the proof for balls centered at the origin.

Using the minimality of  $u$ , we have

$$0 \geq \frac{J(u) - J(u_\varepsilon)}{\varepsilon} = \frac{J(u(x)) - J(u(x + \varepsilon \eta_k(x)x))}{\varepsilon}.$$

Letting  $\varepsilon \rightarrow 0$  this gives

$$\begin{aligned} 0 &\geq \int_{B_r} (|\nabla u|^2 \operatorname{div}(\eta_k(x)x) - 2\nabla u D(\eta_k(x)x) \nabla u) \\ &= \int_{B_r} ((n-2)|\nabla u|^2 \eta_k(x) + |\nabla u|^2 x \nabla \eta_k(x) - 2(x \nabla u)(\nabla u \nabla \eta_k(x))). \end{aligned}$$

Sending this time  $k \rightarrow \infty$ , we obtain

$$0 \geq \int_{B_r} (n-2)|\nabla u|^2 - \int_{\partial B_r} (|\nabla u|^2 x \nu + 2(x \nabla u)(\nu \nabla u)),$$

which is equivalent to (3.10).  $\square$

We can now prove the monotonicity of Almgren's frequency.

*Proof of Theorem 3.1.* The three lemmas proved above imply

$$\frac{N'(r)}{N(r)} \geq \frac{1}{r} + \frac{n-2}{r} - \frac{n-1}{r} + 2 \left( \frac{\int_{\partial B_r(x)} u_\nu^2}{\int_{\partial B_r(x)} u u_\nu} - \frac{\int_{\partial B_r(x)} u u_\nu}{\int_{\partial B_r(x)} u^2} \right) \geq 0.$$

The last inequality follows from the Cauchy-Schwartz inequality, the equality case of which has to be satisfied if  $N'(r) = 0$  and provides that  $u$  is homogeneous (see [GP09] or [ACS08]). From the scaling properties of  $N(r, u)$  we can also see that it is constant when the function  $u$  is homogeneous, thus the theorem is proved.  $\square$

#### 4. BLOWUPS AND POSSIBLE HOMOGENEITIES

**4.1. Blowups.** An important tool for us will be the following rescaling of the minimizers at some points  $x_0 \in \Gamma(u)$ :

$$(4.1) \quad u_r(x) = u_{x_0, r}(x) := \frac{u(rx + x_0)}{\left( \frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u^2 \right)^{1/2}}.$$

The limits of the rescaled functions  $\{u_r\}$  as  $r = r_j \rightarrow 0+$  will be called *blowups* of  $u$  at the point  $x_0$ . The above definition normalizes the  $L^2(\partial B_1)$  norm of the rescaled functions to be one:

$$(4.2) \quad \int_{\partial B_1} u_r^2 = 1.$$

Another useful property is the following identity

$$(4.3) \quad N(\rho, u_r) = N^{x_0}(\rho r, u).$$

We next want to let  $r = r_j \rightarrow 0$  and study the convergence of the rescaled functions  $u_{r_j}$ . We start by showing that such convergence will be strong in  $W^{1,2}$ .

**Lemma 4.1** (Strong convergence). *Let  $u_j$  be a minimizer of  $J$  over  $\mathcal{K}_0(g_j)$  with some  $g_j \in L^2((\partial B_1)^+)$ . Let also  $\|u_j\|_{W^{1,2}(B_1)} \leq C$ , and  $u_j \rightharpoonup u_0$  weakly in  $W^{1,2}(B_1)$  and  $u_j \rightarrow u_0$  in  $C_{\text{loc}}^\alpha(B_1)$ . Then  $u_j \rightarrow u_0$  strongly in  $W_{\text{loc}}^{1,2}(B_1)$ :*

$$(4.4) \quad \int_{B_\rho} |\nabla u_j|^2 \rightarrow \int_{B_\rho} |\nabla u_0|^2 \quad \text{for all } 0 < \rho < 1.$$

Moreover  $u_0$  minimizes  $J$  over  $\mathcal{K}_0(g_0)$  with boundary values  $g_0 = \lim_{j \rightarrow \infty} g_j$ .

*Proof.* 1) We first prove that for any two solutions  $u_1$  and  $u_2$ ,  $(u_2 - u_1)_\pm$  are subharmonic:

$$(4.5) \quad \int_{B_1} \nabla(u_2 - u_1)_\pm \nabla \eta \leq 0$$

for all nonnegative test functions  $\eta \in C_0^\infty(B_1)$ . We will show only the subharmonicity of  $(u_2 - u_1)_+$ , the other one being analogous. Now since the only complications can occur on  $B'_1 \cap \{x_1 > 0\}$ , without loss of generality we may assume that  $E = \{u_2 > u_1\} \cap B'_1 \subset B'_1 \cap \{x_1 > 0\}$  is nonempty. Then from the Signorini conditions on  $B'_1$  we have that

$$\partial_{x_n} u_2 = 0 \quad \text{on } E, \quad \partial_{x_n} u_1 \leq 0 \quad \text{on } E, \quad \partial_{x_n}(u_2 - u_1) \geq 0 \quad \text{on } E.$$

For any point  $x_0 \in E$ , let  $\delta > 0$  be such that  $B'_\delta(x_0) \subset E$ . Then from harmonicity of  $u_2 - u_1$  in  $B_1^\pm$ , we have that for any test function  $\eta \geq 0$ ,  $\eta \in C_0^\infty(B_\delta(x_0))$ ,

$$\int_{B_\delta(x_0)} \nabla(u_2 - u_1) \nabla \eta = 2 \int_{B_\delta^+(x_0)} \nabla(u_2 - u_1) \nabla \eta = - \int_{B_\delta^+(x_0)} \partial_{x_n}(u_2 - u_1) \eta \leq 0.$$

This implies the subharmonicity of  $(u_2 - u_1)_+$  in a neighborhood of any point  $x_0 \in E$ , implying the subharmonicity in  $B_1$ .

2) Take the sequence  $\{u_j\}$  and  $u_0$  as in the statement of the lemma. The previous step shows that  $(u_j - u_k)_\pm$  are subharmonic. Letting  $k \rightarrow \infty$  we get  $(u_j - u_0)_\pm$  is also subharmonic. Now using the energy inequality we obtain

$$(4.6) \quad \int_{B_\rho} |\nabla(u_j - u_0)_\pm|^2 \leq C(\rho) \int_{B_1} (u_j - u_0)_\pm^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which implies the strong convergence in  $B_\rho$ .

3) Recall now that  $u_j$  minimizes  $J$  over  $\mathcal{K}_0(g_j)$ . Since  $u_j$  are bounded in  $W^{1,2}(B_1)$  and the trace mapping is compact, we can take a subsequence such that  $g_j \rightarrow g_0$  in  $L^2(\partial B_1)$  as  $j \rightarrow \infty$ . Taking the minimizer  $\hat{u}_0$  of  $J$  on  $\mathcal{K}_0(g_0)$  and letting  $u_0$

be the strong limit of  $u_j$  obtained in previous step and using that  $(u_j - \hat{u}_0)_\pm$  is subharmonic, we obtain

$$(4.7) \quad \sup_{B_\rho} |(u_j - \hat{u}_0)_\pm| \leq C(\rho) \int_{\partial B_1^+} (g_j - g_0)_\pm^2.$$

Thus  $u_j$  converges uniformly to  $\hat{u}_0$  on  $B_\rho$  for any  $0 < \rho < 1$ , meaning  $\hat{u}_0 \equiv u_0$  in  $B_1$ .  $\square$

**4.2. Homogeneity of blowups.** We next show that the blowups are homogeneous.

**Lemma 4.2** (Homogeneity of blowups). *Let  $u$  be a minimizer of  $J$  over  $\mathcal{K}_0$  and  $u_0 = \lim_{r_j \rightarrow 0} u_{r_j}$  to be a blowup of  $u$  at  $x_0 \in \Gamma(u)$ . Then  $u_0$  is homogeneous of degree  $\kappa = N^{x_0}(0+, u)$ .*

*Proof.* Indeed, using (4.3) and Theorem 3.1 for a  $0 < r < 1/2$  we obtain

$$N(1, u_r) = N(r, u) \leq N(1/2, u) =: M.$$

Using (4.2) and the above estimate we arrive at

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) \leq M,$$

which shows that the sequence  $\{u_r\}$  is bounded in  $W^{1,2}(B_1)$ . Thus, we can choose a weakly converging subsequence  $u_{r_j} \rightharpoonup u_0$  in  $W^{1,2}(B_1)$ . From Lemma 4.1 we also have the strong convergence  $u_{r_j} \rightarrow u_0$  in  $W_{\text{loc}}^{1,2}(B_1)$ , which means in particular that

$$(4.8) \quad \lim_{r_j \rightarrow 0} N(\rho, u_{r_j}) = N(\rho, u_0),$$

provided  $\int_{\partial B_\rho} u_0^2 \neq 0$ . Now suppose  $\int_{\partial B_\rho} u_0^2 = 0$ . Then by the maximum principle we would have that the subharmonic functions  $(u_0)_\pm$  vanish in  $B_\rho$ , and since  $u_0$  is harmonic in  $B_1^+$ , we obtain that  $u_0 \equiv 0$  in  $B_1$ . But due to compactness of trace mapping, we have

$$\int_{\partial B_1} u_0^2 = \lim_{r_j \rightarrow 0} \int_{\partial B_1} u_{r_j}^2 d\sigma = 1,$$

which contradicts to  $u_0$  vanishing in  $B_1$ . Thus (4.8) holds for any  $0 < \rho < 1$ . Moreover we can write

$$N(\rho, u_0) = \lim_{r_j \rightarrow 0} N(\rho, u_{r_j}) = \lim_{r_j \rightarrow 0} N^{x_0}(\rho r_j, u) = N^{x_0}(0+, u) =: \kappa,$$

yielding

$$(4.9) \quad N(\rho, u_0) \equiv \kappa \quad \text{for any } 0 < \rho < 1.$$

Then using the last part of Theorem 3.1, we complete the proof of the lemma.  $\square$

We can now proceed to the proof of Theorems 1.2 and 1.3.

### 4.3. Minimal homogeneity at contact points.

*Proof of Theorem 1.2.* For a fixed  $r > 0$  consider a functional

$$(4.10) \quad \Gamma(u) \ni x \mapsto N^x(r, u) = \frac{r \int_{B_r(x)} |\nabla u|^2}{\int_{\partial B_r(x)} u^2}.$$

Then, since for a fixed  $r > 0$  the functional above is continuous and that  $N^x(r, u)$  is nondecreasing in  $r$ , we obtain the upper semicontinuity of the functional  $x \mapsto N^x(0+, u)$  on  $\Gamma(u)$ . More precisely, we have

$$(4.11) \quad N^{x_0}(0+, u) \geq \limsup_{\substack{x \rightarrow x_0 \\ x \in \Gamma(u)}} N^x(0+, u).$$

Now, for a contact point  $\bar{x} \in \Gamma'(u)$  we have a sequence of free boundary points  $x_j \in \Gamma(u) \cap \{x_1 > 0\}$  converging to  $\bar{x}$ . Now, near  $x_j$ , the minimizer  $u$  solves the Signorini problem and therefore we have

$$N^{x_j}(0+, u) \geq 3/2, \quad \text{for } x_j \in \Gamma(u) \cap \{x_1 > 0\}$$

and thus, using the upper semicontinuity, we conclude that

$$N^{\bar{x}}(0+, u) \geq 3/2. \quad \square$$

### 4.4. Possible homogeneities at non-contact points.

*Proof of Theorem 1.3.* Since  $\bar{x}$  is not a contact point, we know that there exists a positive  $\delta$  such that  $u$  is harmonic in  $B_\delta(\bar{x}) \setminus (B'_\delta(\bar{x}) \cap \{x_1 \leq 0\})$ . Let  $u_0$  be a blowup of  $u$  at  $\bar{x}$ :

$$u_0 = \lim_{r_j \rightarrow 0} u_{\bar{x}, r_j} = \lim_{r_j \rightarrow 0} \frac{u(\bar{x} + r_j x)}{\left( \frac{1}{r_j^{n-1}} \int_{\partial B_{r_j}(\bar{x})} u^2 d\sigma \right)^{1/2}}.$$

We know that  $u_0$  is homogeneous of degree  $\kappa = \kappa(\bar{x}) := N^{\bar{x}}(0+, u)$ , meaning  $u_0(r\theta) = r^\kappa u_0(\theta)$  for  $r > 0$  and  $\theta \in \partial B_1$ . We also know that  $u_0$  is harmonic in  $\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \cap \{x_1 \leq 0\})$  and  $u_0$  is nonnegative in  $\mathbb{R}^{n-1} \cap \{x_1 > 0\}$ . Next, for  $m \in \mathbb{N}$ , define

$$(4.12) \quad \hat{u}_{m-1/2}(x) := \operatorname{Re}(x_1 + i|x_n|)^{m-1/2}.$$

It is easy to see that  $\hat{u}_{m-1/2}$  is homogeneous of degree  $(m-1/2)$  and

$$\begin{aligned} \Delta \hat{u}_{m-1/2} &= 0 \quad \text{in } \mathbb{R}^n \setminus (\mathbb{R}^{n-1} \cap \{x_1 \leq 0\}) \\ \hat{u}_{m-1/2} &= 0 \quad \text{on } \mathbb{R}^{n-1} \cap \{x_1 \leq 0\}. \end{aligned}$$

Thus, the set of possible values of  $\kappa$  includes  $\{m-1/2 : m \in \mathbb{N}\}$ . We want to show that those are the only possible values of  $\kappa$ . This fact will follow from the expansion of harmonic functions in slit domains, recently established in [DSS14, Theorem 3.1]. The latter theorem implies that for any  $k \geq 0$  there exists a polynomial  $P_0(x, r)$  of degree  $k+1$  such that

$$u_0(x) = \hat{u}_{1/2}(x) (P_0(x', r) + o(|x|^{k+1})), \quad r = \sqrt{x_1^2 + x_n^2},$$

solely from the fact that  $u_0$  is harmonic in  $B_1 \setminus B'_1 \cap \{x_1 \leq 0\}$ , vanishes continuously on  $B'_1 \cap \{x_1 \leq 0\}$  and is even in  $x_n$ . Taking  $k > \kappa$  and using that  $u_0$  is homogeneous of degree  $\kappa$ , we obtain that

$$u_0(x) = \hat{u}_{1/2}(x)P_0(x', r)$$

for a homogeneous polynomial  $P_0(x', r)$  of degree  $\kappa - 1/2$ . Thus,  $\kappa = m - 1/2$  for some  $m \in \mathbb{N}$ . The proof is complete.  $\square$

## REFERENCES

- [Alm00] Frederick J. Almgren Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, vol. 1, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.  $Q$ -valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2; With a preface by Jean E. Taylor and Vladimir Scheffer. MR1777737 (2003d:49001)
- [AU95] D. E. Apushkinskaya and N. N. Ural'tseva, *On the behavior of the free boundary near the boundary of the domain*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **221** (1995), no. Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 26, 5–19, 253, DOI 10.1007/BF02355579 (Russian, with English and Russian summaries); English transl., J. Math. Sci. (New York) **87** (1997), no. 2, 3267–3276. MR1359745 (96m:35340)
- [AC04] I. Athanasopoulos and L. A. Caffarelli, *Optimal regularity of lower dimensional obstacle problems*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **310** (2004), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34], 49–66, 226, DOI 10.1007/s10958-005-0496-1 (English, with English and Russian summaries); English transl., J. Math. Sci. (N. Y.) **132** (2006), no. 3, 274–284. MR2120184 (2006i:35053)
- [ACS08] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa, *The structure of the free boundary for lower dimensional obstacle problems*, Amer. J. Math. **130** (2008), no. 2, 485–498, DOI 10.1353/ajm.2008.0016. MR2405165 (2009g:35345)
- [Caf79] L. A. Caffarelli, *Further regularity for the Signorini problem*, Comm. Partial Differential Equations **4** (1979), no. 9, 1067–1075, DOI 10.1080/03605307908820119. MR542512 (80i:35058)
- [Caf98] ———, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), no. 4-5, 383–402, DOI 10.1007/BF02498216. MR1658612 (2000b:49004)
- [CSS08] Luis A. Caffarelli, Sandro Salsa, and Luis Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425–461, DOI 10.1007/s00222-007-0086-6. MR2367025 (2009g:35347)
- [DSS14a] Daniela De Silva and Ovidiu Savin,  *$C^\infty$  regularity of certain thin free boundaries* (2014), preprint, available at [arXiv:1402.1098](https://arxiv.org/abs/1402.1098).
- [DSS14b] ———, *Boundary Harnack estimates in slit domains and applications to thin free boundary problems* (2014), preprint, available at [arXiv:1406.6039](https://arxiv.org/abs/1406.6039).
- [GP09] Nicola Garofalo and Arshak Petrosyan, *Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem*, Invent. Math. **177** (2009), no. 2, 415–461, DOI 10.1007/s00222-009-0188-4. MR2511747 (2010m:35574)
- [KPS14] Herbert Koch, Arshak Petrosyan, and Wenhui Shi, *Higher regularity of the free boundary in the elliptic Signorini problem* (2014), preprint, available at [arXiv:1406.5011](https://arxiv.org/abs/1406.5011).
- [LS67] J.-L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–519. MR0216344 (35 #7178)
- [PSU12] Arshak Petrosyan, Henrik Shahgholian, and Nina Uraltseva, *Regularity of free boundaries in obstacle-type problems*, Graduate Studies in Mathematics, vol. 136, American Mathematical Society, Providence, RI, 2012. MR2962060
- [Ric78] David Joseph Allyn Richardson, *Variational problems with thin obstacles*, ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)—The University of British Columbia (Canada). MR2628343
- [Ura87] N. N. Ural'tseva, *On the regularity of solutions of variational inequalities*, Uspekhi Mat. Nauk **42** (1987), no. 6(258), 151–174, 248 (Russian). MR933999 (90c:35033)

- [Ura96] ———,  *$C^1$  regularity of the boundary of a noncoincident set in a problem with an obstacle*, Algebra i Analiz **8** (1996), no. 2, 205–221 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **8** (1997), no. 2, 341–353. MR1392033 (97m:35105)
- [Wei98] Georg S. Weiss, *Partial regularity for weak solutions of an elliptic free boundary problem*, Comm. Partial Differential Equations **23** (1998), no. 3–4, 439–455, DOI 10.1080/03605309808821352. MR1620644 (99d:35188)

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